Stabilization of SDEs driven by G-Brownian motion

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- The scalar linear stochastic differential equation

$$dx(t) = ax(t)dt + \sum_{i=1}^{m} b_i x(t) dB_i(t), t \geq t_0.$$

- The sample Lyapunov exponent is

$$\lim_{t\to\infty} \frac{1}{t} \log|x(t;t_0,x_0)| = a - \frac{1}{2}\sum_{i=1}^{m} b_i^2, \text{a.s.}$$

- Li et al. (2016) proposed the sufficient conditions for the exponentially instability to G-SDEs with the following form

$$\begin{aligned} \mathrm{d}x(t) &= f(t,x(t))\mathrm{d}t + h(t,x(t))\mathrm{d}\langle B\rangle(t) \\ &\quad +\sigma(t,x(t))\mathrm{d}B(t), \quad t \geq 0. \end{aligned}$$

- h–be viewed as mean-uncertainty perturbation
- σ– be viewed as volatility-uncertainty perturbation
- A natural question is whether we can design a controller to make the system be stable.

Content

1. Non linear expectations: a general framework

Setting

- Let $C_{l,Lip}(\mathbb{R})$ denote the space of all local Lipschitz functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, satisfying that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad x, y \in \mathbb{R}^n,$$

  for some $C > 0, \quad m \in \mathbb{N}$ depending on φ.

- Let $\Omega = \mathbb{R}$ and let $\mathcal{H}$ be a linear space of real functions s.t.

$$X_1, X_2, \cdots, X_n \in \mathcal{H} \Rightarrow \varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}, \ \forall \varphi \in C_{l,Lip}(\mathbb{R}).$$

1. Non linear expectations: a general framework

Definition

A non linear expectation $\widehat{\mathbb{E}}$ is a functional $\mathcal{H} \to \mathbb{R}$, for $X, Y \in \mathcal{H}$, satisfying the following properties:

- **Monotonicity** : if $X \geq Y$, then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$.
- **Constant preservation** : $\widehat{\mathbb{E}}[c] = c$ for $c \in \mathbb{R}$.
- **Subadditivety** : $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ or $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$.
- **Positive homogeneity** : $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}$ for $\lambda \geq 0$.

Remark

Norm on $\mathcal{H}$: $||X|| := \widehat{\mathbb{E}}|X|, X \in \mathcal{H}$. $(\mathcal{H}, ||\cdot||)$ is a normed space.

2. G-Normal distribution

Let $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2, X \in \mathcal{H}, G(a) = \frac{1}{2}(a^+\overline{\sigma}^2 - a^-\underline{\sigma}^2), a \in \mathbb{R}$.

G-case (Definition)

$X \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ is characterized by $u(t,x) = \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, for $\varphi \in C_{l,Lip}(\mathbb{R})$ which is the unique viscosity solution of

$$\partial_t u - G(\partial^2_{xx} u) = 0, \; u(0,x) = \varphi(x).$$

Classical case

If $\sigma^2 = \overline{\sigma}^2 = \underline{\sigma}^2, X \sim N(0, \sigma^2)$ is characterized by $u(t,x) = \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, for $\varphi \in C_{l,Lip}(\mathbb{R})$ which satisfies that

$$\partial_t u = \frac{1}{2}\sigma^2 \partial^2_{xx} u(t,x), \; u(0,x) = \varphi(x).$$

3. G-Brownian motion under G-expectation

Setting

- Let Ω denote the space of all $\mathbb{R}$–valued continuous functions with $\omega_0 = 0$, equipped with the distance
$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|\right) \wedge 1\right].$$
- Canonical process on Ω : $B_t(\omega) = \omega_t, t \geq 0$.
- For $t > 0$, we set

$$\mathcal{H}_t := \{\varphi(\omega_{s_1}, \omega_{s_2}, \cdots, \omega_{s_n}), n > 1,$$
$$s_1, s_2, \cdots, s_n \in [0, t], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}.$$

Set

$$\mathcal{H} := \bigcup_{i=1}^{\infty} \mathcal{H}_n.$$

3. G-Brownian motion under G-expectation

Definition

The canonical process $(B_t)_{t\geq 0}$ on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called as a G-Brownian motion if

(1) $\textbf{Initial value}: B_0(\omega) = 0;$

(2) $\textbf{Distribution}: B_{t+s} - B_t$ is $N(0, [\overline{\sigma}^2 s, \underline{\sigma}^2 s])$ distributed for $s, t \geq 0;$

(3) $\textbf{Increment independent}:$ for $n \geq 2, 0 \leq t_1 \leq \cdots \leq t_n, B_{t_n} - B_{t_{n-1}}$ is independent of $(B_{t_1}, B_{t_2}, \cdots, B_{t_{n-1}})$.

3. G-Brownian motion under G-expectation

Theorem

Let $(B_t(\omega))_{t\geq 0}$ be a process on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ such that

(1) For each $0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent of $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$.

(2) B_t has the same distribution as $B_{t+s} - B_t$ for $s, t \geq 0$;

(3) $\lim_{t\downarrow 0} \widehat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$,

then B is a G-Brownian motion, for $\overline{\sigma}^2 = -\widehat{\mathbb{E}}[-B_1^2], \underline{\sigma}^2 = \widehat{\mathbb{E}}[B_1^2]$.

4. Itô's integral of G-Brownian motion

Setting

- $L_G^2(\mathcal{H}_t) := \{\xi \in \mathcal{H}_t : \widehat{\mathbb{E}}|\xi|^2 < \infty\}$.
- $L_G^{2,0}([0,T])$ of simple processes by

$$L_G^{2,0}([0,T]) := \Big\{\eta_t(\omega) := \sum_{j=1}^{N-1} \xi_{t_j}(\omega) I_{[t_j,t_{j+1})}; \xi_{t_j}(\omega) \in L_G^2(\mathcal{H}_{t_j}), 0 = t_0 < t_1 < \cdots < t_N = T\Big\}.$$

4. Itô's integral of G-Brownian motion

Definition

For $\eta_t(\omega) = \sum_{j=1}^{N-1} \xi_{t_j}(\omega) I_{[t_j, t_{j+1})} \in L_G^{p,0}([0,T])$, we define the Itô's integral by $I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=1}^{N-1} \xi_{t_j} \left(B_{t_{j+1}\wedge t} - B_{t_j \wedge t} \right)$.

Property

- $\widehat{\mathbb{E}} \int_0^T \eta_t dB_t = 0, \quad \widehat{\mathbb{E}} \left(\int_0^T \eta_t dB_t \right)^2 \leq \int_0^T \widehat{\mathbb{E}}(\eta_t)^2 dt.$

$L_G^2([0,T])$ denotes the completion of $L_G^{2,0}([0,T])$ under the norm $|\eta|_{L_G^2([0,T])} = \left(\int_0^T \widehat{\mathbb{E}}(\eta_t)^2 dt \right)^{1/2}$. Hence, the stochastic integral can be extended to $L_G^2([0,T])$.

4. Itô's integral of G-Brownian motion

Quadratic Variation Process

The quadratic variation process of the G-Brownian motion B by

$$\langle B\rangle_t := \lim_{N\to\infty}\sum_{j=1}^{N-1}\left(B_{t_{j+1}^N} - B_{t_j^N}\right)^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

Property

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^T (\eta_t)^2 d\langle B\rangle_t\right], \eta \in L_G^2([0,T]).$$

5. Developments on Itô's calculus based on G-B.M. and G-SDE

- Peng (2007a, 2007b) established the fundamental theory of G-expectation, Itô stochastic calculus based on G-expectation.
- Denis et al. (2011) gave the application on risk dynamics based on G-B.M..
- Gao (2009, 2010) proved the pathwise properties and homeomorphic property with respect to the initial values and established the large deviation principle for G-SDE.
- Luo et al. (2014, 2016) established the relation between G-SDE and ODE and studied a class of reflected G-SDEs with nonlinear resistance.

- Zhang et al. (2011, 2012) considered the exponential stability for G-SDE and investigated the stochastic optimal control problems under G-expectation.
- Lin (2013) proved the existence and uniqueness for a class of reflected G-SDEs.
- Li et al. (2016) studied the solvability and stability for G-SDE with Lyapunov-type conditions by localization methods.
- Hu, Ji, Peng, Song et al. in series works discussed backward SDE driven by G-B.M. and its applications in stochastic recursive optimal control problem an so on.

- Ren et al. (2011, 2013, 2015, 2016a, 2016b, 2017a-2017d, 2018a-2018e)
  - stability for some classes of (impulsive) G-SDE (neural networks)
  - the existence, uniqueness and stability for G-SDE with infinite delay
  - the square-mean pseudo almost automorphic mild solutions for stochastic evolution equations driven by G-Brownian motion
  - the multi-valued G-SDE and its related optimal control
  - stabilization of G-SDE and applications

6-1. Stabilization of G-SDEs (feedback control)

We consider the following G-SDE

$$\begin{aligned} \mathrm{d}x(t) &= f(t,x(t))\mathrm{d}t + h(t,x(t))\mathrm{d}\langle B\rangle(t) \\ &\quad +\sigma(t,x(t))\mathrm{d}B(t),\ \ t \geq 0, \end{aligned} \tag{3.1}$$

- the initial data $x(0) = x_0 \in \mathrm{R}^n$, $B(\cdot)$ a one dimensional G-Brownian motion, $\langle B\rangle(\cdot)$ the quadratic variation process of the G-BM $B(\cdot)$.
- $f, h, \sigma : \mathrm{R}^+ \times \mathrm{R}^n \to \mathrm{R}^n$ and $f, h, \sigma \in M_G^2(0,T)$.

- **Theorem 1** (Li et al. (2016)) For the system (3.1), whose coefficients are deterministic functions and satisfy the Lipschitz condition in x. If there exists a function $V(t,x) \in C^{1,2}(\mathrm{R}^+ \times \mathrm{R}^n; \mathrm{R}^+)$ and three positive constants c_1, c_2 and q such that

$$c_1|x(t)|^p \leq V(t,x(t)) \leq c_2|x(t)|^p$$

and

$$LV(t,x(t)) \geq \eta|x(t)|^q.$$

Then, the trivial solution of system (3.1) is qth exponentially instable, that is,

$$\mathbb{E}|x(t)|^q \geq C|x_0|^q e^{\eta t}.$$

- If the system (3.1) is unstable, we design a controller $u(t, x(t))$ in the drift part so that the system becomes stable.
- The controlled system is expressed as

$$\begin{aligned} \mathrm{d}x(t) &= [f(t, x(t)) + u(t, x(t))]\mathrm{d}t + h(t, x(t))\mathrm{d}\langle B\rangle(t) \\ &\quad +\sigma(t, x(t))\mathrm{d}B(t), \;\; t \geq 0. \end{aligned} \tag{3.2}$$

- For stability analysis, we assume that $x_0 = 0, f(t, 0) \equiv 0, u(t, 0) \equiv 0, h(t, 0) \equiv 0, \sigma(t, 0) \equiv 0$ for all $t \geq 0$, then the system admits a trivial solution $x(t) \equiv 0$.

- Let $C^{1,2}(\mathrm{R}^+ \times \mathrm{R}^n; \mathrm{R}^+)$ be the family of all nonnegative functions $U(t,x)$ on $\mathrm{R}^+ \times \mathrm{R}^n$, which once differentiable are continuous in t and twice differentiable in x.
- If $U \in C^{1,2}(\mathrm{R}^+ \times \mathrm{R}^n; \mathrm{R}^+)$, we define an operator LU as follows

$$LU(t,x) := U_t(t,x) + U_x(t,x)[f(t,x) + u(t,x)]$$

$$+G\left(\langle U_x(t,x), 2h(t,x)\rangle + \langle U_{xx}(t,x)\sigma(t,x), \sigma(t,x)\rangle\right),$$

where

$$U_t(t,x) = \frac{\partial U(t,x)}{\partial t},$$

$$U_x(t,x) = \left(\frac{\partial U(t,x)}{\partial x_1}, \frac{\partial U(t,x)}{\partial x_2}, \cdots, \frac{\partial U(t,x)}{\partial x_n}\right),$$

$$U_{xx}(t,x) = \left(\frac{\partial^2 U(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

- **Theorem 2** The system (3.2) is pth moment exponentially stable ($\mathbb{E}|x(t,x_0)|^p \leq M|x_0|^p e^{-\lambda t}, t \geq 0$) if there exists a positive constant λ such that the function $U(t,x(t))$ and $LU(t,x(t))$ satisfy the following conditions.

(A1) There are two positive constants ρ_1 and ρ_2 such that

$$\rho_1|x(t)|^p \leq U(t,x(t)) \leq \rho_2|x(t)|^p. \tag{3.3}$$

(A2) There exists a positive constant λ such that

$$LU(t,x(t)) \leq -\lambda U(t,x(t)). \tag{3.4}$$

6-2. An example

We discuss the following one-dimensional G-SDE,

$$\begin{aligned} \mathrm{d}x(t) &= \frac{1}{2}x(t)\cos t\mathrm{d}t + \left(\frac{\pi}{2} + \frac{1}{2}\arctan t\right) x(t)\mathrm{d}\langle B\rangle(t) \\ &\quad + \sqrt{\pi + \arctan t}x(t)\mathrm{d}B(t), t \geq 0, \end{aligned} \tag{3.5}$$

where $B(t)$ is one dimensional G-Brownian motion and $B(t) \sim N(0, [\frac{1}{\pi}, \frac{2}{\pi}])$. Let $U = |x|^2$. We have

$$U_x(t, x(t))f(t, x(t)) = |x(t)|^2 \cos t,$$
$$U_x(t, x(t))h(t, x(t)) = 2|x(t)|^2 \left(\frac{\pi}{2} + \frac{1}{2}\arctan t\right),$$
$$U_{xx}(t, x(t))\sigma^2(t, x(t)) = 2|x(t)|^2(\pi + \arctan t)$$

$$
\begin{aligned}
& G\left(\langle U_x(t,x), 2h(t,x)\rangle + \langle U_{xx}(t,x)\sigma(t,x), \sigma(t,x)\rangle\right) \\
= \;& G\left(2|x|^2(\pi + \arctan t) + 2|x|^2(\pi + \arctan t)\right) \\
= \;& G\left(4|x|^2(\pi + \arctan t)\right) \\
\geq \;& G\left(2\pi|x|^2\right) \\
= \;& 2|x|^2.
\end{aligned}
$$

$$
\begin{aligned}
LU(t,x(t)) &= 2x(t)f(t,x(t)) + G\big(\langle U_x(t,x), 2h(t,x)\rangle \\
&\quad + \langle U_{xx}(t,x)\sigma(t,x), \sigma(t,x)\rangle\big) \\
&\geq -|x(t)|^2 + 2|x(t)|^2 \\
&= |x(t)|^2.
\end{aligned}
$$

From **Theorem 1**, the system (3.5) is not mean square exponentially stable.

Let us design a linear feedback controller $u(t, x(t)) = kx(t)$, where k is a constant. The controlled system has the form

$$\begin{aligned} \mathrm{d}x(t) &= \left[\frac{1}{2}x(t)\cos t + kx(t)\right]\mathrm{d}t \\ &\quad + \left(\frac{\pi}{2} + \frac{1}{2}\arctan t\right)x(t)\mathrm{d}\langle B\rangle(t) \\ &\quad + \sqrt{\pi + \arctan t}\,x(t)\mathrm{d}B(t), t \geq 0. \end{aligned} \tag{3.6}$$

Letting $k=-5$, we have

$$
\begin{aligned}
LU(t,x(t)) &= U_x(t,x)[f(t,x(t))+u(x(t))] \\
&\quad +G\left(\langle U_x(t,x),2h(t,x)\rangle+\langle U_x(t,x)\sigma(t,x),\sigma(t,x)\rangle\right) \\
&= 2x(t)\left[\frac{1}{2}x(t)\cos t-5x(t)\right] \\
&\quad +G\left(2|x(t)|^2(\pi+\arctan t)+2|x(t)|^2(\pi+\arctan t)\right) \\
&\leq |x(t)|^2-10|x(t)|^2+G(6\pi|x(t)|^2) \\
&= -9|x(t)|^2+\frac{1}{2}\cdot\frac{2}{\pi}\cdot 6\pi|x(t)|^2 \\
&= -3|x(t)|^2.
\end{aligned}
$$

From **Theorem 2**, the controlled system (3.6) is mean square exponentially stable.

6-3. Stabilization of G-SDEs (discrete-time state observation)

We consider the following G-SDE

$$\begin{aligned} \mathrm{d}x(t) &= [f(t,x(t)) + u(t,x(\delta_t))]\mathrm{d}t + h(t,x(t))\mathrm{d}\langle B\rangle(t) \\ &\quad +\sigma(t,x(t))\mathrm{d}B(t),\ t \geq 0, \end{aligned}$$

where $\delta_t = [t/\tau]\tau$ is the integer part of t/τ, τ is the discrete-time observation gap.

6-3. An example

We discuss the following one-dimensional G-SDE,

$$\begin{aligned} \mathrm{d}x(t) &= x(t)\sin t\mathrm{d}t + (2+\sin t)x(t)\mathrm{d}\langle B\rangle(t) \\ &\quad + \sqrt{3+\cos t}x(t)\mathrm{d}B(t), \quad t \geq 0, \end{aligned} \tag{3.7}$$

where $B(t)$ is one dimension G-Brownian motion and $B(t) \sim N(0, [\frac{1}{2}, 1])$.

From Li et al. (2016), the system (3.7) is not mean square exponentially stable.

Let us design a discrete-time linear feedback controller function with the form

$$\begin{aligned} \mathrm{d}x(t) &= [x(t)\sin t - 7x(\delta_t)]\mathrm{d}t + (2+\sin t)x(t)\mathrm{d}\langle B\rangle(t) \\ &\quad + x(t)\sqrt{3+\cos t}\,\mathrm{d}B(t), \quad t \geq 0, \end{aligned} \tag{3.8}$$

with $\tau < 0.000785$.

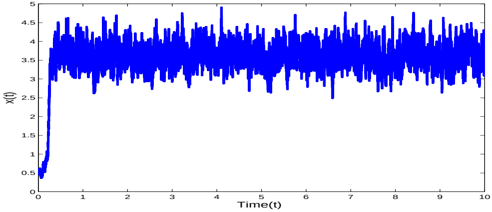

Figure: Response without control

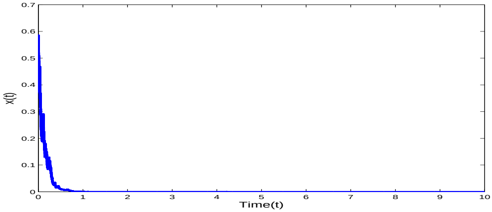

Figure: Response with control

References

- L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes arXiv: 0802.1240v1.
- F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion. Stochastic Process. Appl. 11 (2009) 3356–3382.
- M. Hu, S. Ji, Stochastic maximum principle for stochastic recursive optimal control problem under volatility ambiguity. SIAM J. Control Optim. 54 (2016) 918–945.

- M. Hu, S. Ji, S. Yang, A stochastic recursive optimal control problem under the G-expectation framework. Appl. Math. Optim. 70 (2014), no. 2, 253–278.
- M. Hu, S. Ji, S. Peng, Y. Song, Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. Stochastic Process. Appl. 124 (2014) 1170–1195.
- M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equations driven by G-Brownian motion. Stochastic Process. Appl. 124 (2014)- 759–784
- X. Li, X. Lin, Y. Lin, Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion. J. Math. Anal. Appl. 439 (2016) 235–255.
- Y. Lin, Stochastic differential equations driven by G-Brownian motion with reflecting boundary conditions. Electron. J. Probab. 18 (2013) 1–23.

- P. Luo, Reflcted stochastic differential equations driven by G-Brownian motion with nonlinear resistance. Front. Math. China 11 (2016) 123–140.
- P. Luo, F. Wang, Stochastic differential equations driven by G-Brownian motion and ordinary differential equations. Stochastic Process App. 124 (2014) 3869–3885.
- S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Itô type. Stochastic Process, Appl. Abel Symp, Springer, Berlin. 2 (2007) 541–567.
- S. Peng, G-Brownian motion and dynamic risk measures under volatility uncertainty. arXiv: 0711.2834v1.
- S. Peng, Theory, methods and meaning of nonlinear expectation theory (in Chinese). Sci Sin Math. 47 (2017) 1223–1254.

- Y. Ren, Q. Bi, R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion. Math. Meth. Appl. Sci. 36 (2013) 1746–1759.
- Y. Ren, L. Hu, A note on the stochastic differential equations driven by G-Brownian motion. Statist. Probab. Lett. 81 (2011) 580–585.
- Y. Ren, X. Jia, L. Hu, Exponential stability of solutions to implusive stochastic differential equations driven by G-Brownian motion. Discrete Cont. Dyn-B. 20 (2015) 2157–2169.
- Y. Ren, X. Jia, R. Sakthivel, The p-th moment stability of solutions to implusive stochastic differential equations driven by G-Brownian motion. Appl. Anal. 96 (2017a) 988–1003.
- Y. Ren, J. Wang, L. Hu, Multi-valued stochastic differential equations driven by G-Brownian motion and related stochastic control problems. International Journal of Control 90 (2017) 1132–1154.
- Y. Ren, Y. Gu, Q. Zhou, Stability analysis of impulsive stochastic Cohen-Grossberg neural networks driven by G-Brownian motion, International Journal of Control, to appear (2017c)

- Y. Ren, W. Yin, R. Sakthivel, Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation, Automatic 95 (2018a) 146-151
- Y. Ren, W. Yin, W. Lu, Asymptotical boundedness for stochastic coupled systems on networks driven by G-Brownian motion, Journal of Mathematical Analysis and Applications 466 (2018b) 338-350
- Y. Ren, W. Yin, Asymptotical boundedness for stochastic coupled systems on networks with time-varying delay driven by G-Brownian motion, International Journal of Control, to appear (2018c)
- Y. Ren, W. Yin, D. Zhu, Exponential stability of SDEs driven by G-Brownian motion with delayed impulsive effects: average impulsive interval approach, Discrete and Continuous Dynamical System-B, to appear (2018d)
- H. Yang, Y. Ren, W. Lu, Stabilization of SDEs driven by G-Brownian motion via aperiodically intermittent control, International Journal of Control, to appear (2018e)

- Y. Gu, Y. Ren, R.Sakthivel, Square-mean pseudo almost automorphic mild solutions for stochastic evolution equations driven by G-Brownian motion, Stochastic Analysis and Applications 34 (2016b) 528–545.
- F. Yang, Y. Ren, L. Hu, Multi-valued backward stochastic differential equations driven by G-Brownian motion and its applications, Mathematical Methods in the Applied Sciences 40 (2017b) 4696–4708.
- W. Yin, Y. Ren, Asymptotical boundedness and stability for stochastic differential equations with delay driven by G-Brownian motion, Applied Mathematics Letters 74 (2017) 121–126 (2017d)
- Zhang, D.,Stochastic optimal control problems under G-expectation, Optim. Control Appl. Meth. (2011)

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